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Stäckel potentials and the relevance of the spheroidal coordinates

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Abstract. We study some consequences of the use of spheroidal coordinate frames to describe axisymmetric gravitational fields. In particular, we shall explain the meaning of the third integral of motion, and we shall analyse torque effects measured in a Stäckel potential field.

1. Introduction

The study of axisymmetric potentials is quite relevant to astrophysics, since it recurs in many systems, from stellar to galactic gravitational fields. Orbital motion in such potentials keeps constant both the energy E (due to the time-translational invariance—conservative potentials) and the axial component ℓ of angular momentum (where φ, the azimuthal coordinate, is ignorable). It turns out, moreover, from the analysis of Poincaré sections of the orbits in many realistic galactic potentials, that a third isolating integral should exist [1]. Now, while in spherically symmetric fields this third integral is well known to exist, and its physical meaning is also well known (namely, the square modulus of the total angular momentum), in a general axisymmetric potential it is not clear, *a priori*, when and why such a third integral should be found. As a matter of fact, in the spherically symmetric case the existence of a third integral is linked to the separability of the Hamilton–Jacobi equation, when this is written in 'adapted' (i.e. spherical) coordinates. It is therefore interesting to investigate the separability properties in the axially symmetric case too, testing whether the above machinery producing the desired third integral works in this case as well. It was shown by Stäckel [2] that only a class of axially symmetric potentials, expressed in *spheroidal* coordinates, does allow this separability and thence the additional conserved quantity.

The third integral for axisymmetric fields, when it exists, cannot, however, be identified with the corresponding one in the spherically symmetric case, i.e. it is *not* the square modulus of the angular momentum. Its physical interpretation is still an open question. In this paper we will clarify its structure and explain its meaning. It will be shown how relevant the conformation of the particular coordinate frame used is, in order to obtain the desired physical interpretation.

Dealing with angular momenta and the like, a natural issue would be the analysis of torques, which we will consider as well, in particular, examining the precession induced on the orbital plane by a Stäckel potential and showing how *part* of this effect has to be traced back to a pure choice-of-coordinate cause. It follows that the use of the most convenient coordinate

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frame to describe realistic axisymmetric astrophysical sources, i.e. the spheroidal one, may lead to spurious effects, the presence of which must be taken into account for the results to be not misleading.

2. Separation of the Hamilton–Jacobi equation and Stäckel potentials

Let us consider a potential field $V(x, y, z)$ in Euclidean three-dimensional space; introducing spheroidal coordinates (r, ϑ, φ), which are linked to the Cartesian ones by the relations

$$\begin{aligned} x &= (r^2 + a^2)^{1/2} \sin\vartheta \cos\varphi \\ y &= (r^2 + a^2)^{1/2} \sin\vartheta \sin\varphi \\ z &= r \cos\vartheta \end{aligned} \tag{1}$$

(where a is a parameter describing the oblateness of the coordinate spheroid) the metric tensor turns from the Cartesian form δ_{ij} $(i, j = 1, 2, 3)$ to the (still diagonal) metric g_{ij} whose coefficients are given by

$$\begin{aligned} g_{rr} &= \frac{\Sigma}{r^2 + a^2} \\ g_{\vartheta\vartheta} &= \Sigma \\ g_{\varphi\varphi} &= (r^2 + a^2) \sin^2\vartheta \end{aligned} \tag{2}$$

where we have used the short form $\Sigma = r^2 + a^2 \cos^2\vartheta$. The Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(t, q^i, \frac{\partial S}{\partial q^i}\right) = 0 \qquad \frac{\partial S}{\partial q^i} = p_i \tag{3}$$

for a particle of mass m orbiting in an axisymmetric (and conservative) potential field becomes, in oblate coordinates,

$$\frac{1}{2m}\left[\frac{r^2 + a^2}{\Sigma} p_r^2 + \frac{p_\vartheta^2}{\Sigma} + \frac{\ell^2}{(r^2 + a^2)\sin^2\vartheta}\right] + mV(r, \vartheta) = E \tag{4}$$

where $\ell = p_\varphi$ and $E = -\partial S/\partial t$ are the two already known constants of motion (axial angular momentum and energy). It is now straightforward to see that the separation of equation (4) into two independent equations, one involving just the coordinate r and the other the coordinate ϑ alone, is possible only when the potential term has the form

$$V(r, \vartheta) = \frac{f(r) + g(\vartheta)}{\Sigma} \tag{5}$$

which characterizes the so-called Stäckel potentials†. In equation (5), both $f(r)$ and $g(\vartheta)$ can be thought as 'plain' functions of r and ϑ, i.e. not containing additive constants; this is not restrictive, since an additive constant in the numerator of potential (5) would, in the separation procedure, be simply 'reabsorbed' in the definition of the separation constant K (see later).

† Stäckel potentials are usually given in the form

$$V(u, v) = \frac{F(u) + G(v)}{\sinh^2 u + \cos^2 v}$$

(see, for instance, [1]), where u and v are spheroidal coordinates linked to the usual cylindrical ones (R, z) by

$$R = \Delta \cosh u \sin v \qquad z = \Delta \sinh u \cos v$$

where Δ is a constant. Equation (5) can be recovered if we let $a = \Delta$, $v = \vartheta$ and $r = a \sinh u$.

Here we would like to stress that the Hamilton–Jacobi equation in the axially symmetric case is separable, and thence the third integral exists, *only* when spheroidal coordinates are used *and* the potential belongs to the Stäckel class.

Let us now see whether it is possible to better specify the potential (5) for realistic astrophysical sources: up to this point, in fact, we can just say that equation (5) is the most general form leading to the separation of variables in the presence of axial symmetry. An obvious requirement is a regular behaviour at infinity, i.e. $V(r,\vartheta) \stackrel{r\to\infty}{\longrightarrow} 0$. We can also point out that stars or lenticular galaxies are more similar to spheroids than to other axially symmetric configurations, like (infinite) cylinders; it is then natural to assume that astrophysically useful axial symmetry should bias towards a spheroidal-like structure. We could therefore compare equation (5) with the expression of an exact spheroidal potential, and require that the two share the main features (although they do *not* coincide, since exact spheroidal potentials do *not* allow the separation of variables in the Hamilton–Jacobi equation and hence the third integral we are interested in).

Rewriting equation (5) in *spherical* coordinates R and Θ (connected with the spheroidal ones by the relations $R^2 = r^2 + a^2\sin^2\vartheta$ and $\Theta = \vartheta$), and using $a/R \ll 1$ as the parameter for a series development, we find

$$V(R,\Theta) = \frac{f(R,\Theta)+g(\Theta)}{R^2 - a^2\cos 2\Theta} = \frac{f(R,\Theta)+g(\Theta)}{R^2}\left[1+\left(\frac{a}{R}\right)^2\cos 2\Theta + \mathcal{O}(4)\right]. \tag{6}$$

Now, since $f(R,\Theta)/R^2 \equiv \hat{V}(R,\Theta)$ is an axisymmetric potential, it can always be written in the form

$$\hat{V}(R,\Theta) = -\frac{GM}{R}\left[1-\sum_{n=2}^{\infty}\frac{J_n}{R^n}P_n(\cos\Theta)\right] \tag{7}$$

where the J_n are the multipole coefficients, and the $P_n(\cos\Theta)$ are the Legendre polynomials; in general, these will be of the form $P_{\ell m}(\cos\Theta)$, but in the particular case of axial symmetry the dependence on m disappears and thus $P_n(\cos\Theta) \equiv P_{\ell 0}(\cos\Theta)$. We also note that if symmetry with respect to the equatorial plane is present as well, in the series of equation (7) only those terms having even index differ from zero.

In the axially symmetric case, the multipole coefficients have the following expression:

$$J_n = \frac{1}{(2n+1)M}\int_{\Omega}|\vec{r}'|^n P_n(\cos\Theta)\varrho(\vec{r}')\,\mathrm{d}\vec{r}' \tag{8}$$

where $\varrho(\vec{r})$ is the source mass-density and Ω its volume. We shall thence find that, for instance, J_0 is equal to 1, since the integral in equation (8) for $n = 0$ gives the source mass M; J_1 is identically null, since it defines the centre of mass and we are using a frame in which this is located at the origin; J_2, the quadrupole term, is given by the difference between the polar and equatorial moments of inertia, in units of mass, and so on. In realistic conditions, it is to be expected that $J_n/R^n \ll 1$, at least not in the neighbourhood of the source.

The contribution of $f(R,\Theta)/R^2$ to equation (6) will therefore be of the form

$$\frac{f(R,\Theta)}{R^2}\left[1+\left(\frac{a}{R}\right)^2\cos 2\Theta + \mathcal{O}(4)\right] = -\frac{GM}{R}\left[1-\frac{h(\Theta)}{R^2}+\cdots\right] \tag{9}$$

where the omitted terms are $\mathcal{O}\left(a/R\right)^4 \ll 1$ and $\mathcal{O}\left(J_3/R^3\right) \ll 1$ (or $\mathcal{O}\left(J_4/R^4\right) \ll 1$, if equatorial symmetry is present, which is quite realistic). We then see that (9) has indeed the desired spheroidal-like structure: a monopolar leading term, *no* dipolar term, quadrupolar term, etc.

Let us now see what happens if we consider the contribution of $g(\Theta)/R^2$ in equation (6); this term is also an axisymmetric potential, but obviously it needs no multipole expansion. Since in $g(\Theta)$, as we have previously noted, there are no additive constants, it follows directly that the above term will lead to a queer structure, with *no* monopolar terms, the leading one being *dipolar* (which we should instead expect to be null), *no quadrupolar term* (which should instead be characteristic of a spheroidal-like source) and so on. On these grounds, we therefore require $g(\Theta)$ to be identically zero, for the realistic potentials we are interested in here. Actually, this is not a highly restrictive requirement; such a thing can be shown by considering the following Stäckel potential:

$$\tilde{V}(r, \vartheta) = -\frac{GMr}{\Sigma} \tag{10}$$

obtained from equation (5) with $f(r) = -GMr$ and, indeed, $g(\vartheta) = 0$. In spherical coordinates R and Θ, together with $a \ll R$, it becomes

$$\tilde{V}(R, \Theta) = -\frac{GM}{R}\left[1 - \left(\frac{a}{R}\right)^2 \frac{3\cos^2\Theta - 1}{2}\right] + \mathcal{O}\left(\frac{a}{R}\right)^4. \tag{11}$$

Hence, equation (11) describes (up to terms of $\mathcal{O}\,(a/R)^4 \ll 1$) just the potential generated by a quadrupole (with the very same coefficients, since $P_2(\cos\Theta) = \frac{3}{2}(\cos^2\Theta - 1)$), if only we let a^2 play the role of the gravitational source's quadrupole momentum per unit mass.

The member (10) of the Stäckel class is known as the Keres–Israel potential, and it is quite important, since it constitutes the Newtonian equivalent of the relativistic Kerr solution (see the original articles [3, 4]), faithfully reproducing its characteristics, the only exception being the inertial dragging phenomena (Lense–Thirring effect), which cannot be described by a purely scalar theory as the Newtonian one is†. Given the importance of the Kerr metric in relativistic astrophysics, it is to be expected that its Newtonian counterpart (10) plays an important role in the description of Newtonian axisymmetric potentials, allowing for the existence of the third integral of motion‡. Hence, the condition $g(\Theta) = 0$ turns out to be not groundless.

In the following, for the reasons we have mentioned above, we shall be dealing with Stäckel potentials of the form

$$V(r, \vartheta) = \frac{f(r)}{\Sigma} \tag{12}$$

with such a potential, we will therefore obtain, from equation (4),

$$(r^2 + a^2)p_r^2 - \frac{a^2\ell^2}{(r^2 + a^2)} + 2m^2 f(r) - 2mEr^2 = -K \tag{13}$$

$$p_\vartheta^2 + \frac{\ell^2}{\sin^2\vartheta} - 2mEa^2\cos^2\vartheta = K \tag{14}$$

where K is the separation constant. If we consider a spherically symmetric potential (which implies $a = 0$), we shall directly recognize in the separation constant the square modulus of the total angular momentum. In the more general case we are dealing with, however, this identification is not feasible. While we can still recognize in the first two terms of equation (14) the square of the ϑ- and, respectively, φ-component of the angular momentum, what we need is an explanation for the remaining third term. This we will give in the next section.

† In the relativistic case, a is the Kerr parameter; squared, it can thus be found to have a non-relativistic counterpart in the quadrupole momentum per unit mass of a spheroidal Newtonian source.

‡ Here we note *en passant* that the relativistic metric also has an additional conserved quantity; see, for instance, [5–11].

3. The third integral and its physical meaning

Let us focus our attention on equation (14), which we will rewrite as follows:

$$p_\vartheta^2 + \frac{\ell^2}{\sin^2\vartheta} + 2mEa^2\sin^2\vartheta = \Lambda \tag{15}$$

through a redefinition of the separation constant as

$$\Lambda = K + 2mEa^2 \tag{16}$$

and consider the third term in equation (15). If $E > 0$ the corresponding orbital motion is unbound and we can analyse what happens to this term when $r \to \infty$. Since, as already noted, plausible astrophysical potentials have a regular behaviour at infinity, there the energy will be kinetic alone, $E_\infty = |p_\infty|^2/2m$; but

$$|p_\infty|^2 = \lim_{r\to\infty}(g^{ij}p_i p_j) \tag{17}$$

and in spheroidal coordinates (1) it is easy to show that

$$|p_\infty|^2 \equiv |p_r|^2. \tag{18}$$

Hence, in asymptotic conditions the contribution to E comes from the radial term alone†, and the third term in equation (15) becomes

$$2mEa^2\sin^2\vartheta \overset{r\to\infty}{\longrightarrow} |p_r|^2 a^2 \sin^2\vartheta. \tag{19}$$

This can be shown straightforwardly to be the angular momentum generated purely by radial motion. It is not obvious *a priori*, of course, that such a motion should engender a contribution to angular momentum, but, as we shall now see, that is due to the geometrical features of the coordinate frame employed. (The forthcoming explanation follows the path given in [11] to analyse the meaning of the fourth constant in the Kerr metric; here, however, additional remarks and clarifications to that discussion will be included.)

If we consider the ϑ = constant surfaces of the metric (2), we observe that they have the structure of hyperboloids of rotation around the symmetry axis, orthogonally crossing the $z = 0$ plane into a circle of radius $a\sin\vartheta$. For a given ϑ, the radial coordinate direction at infinity coincides with that of the asymptotes of the hyperboloid with the same ϑ; the radial momentum $\vec{p}_r$ of the orbiting particle at infinity therefore lies in the tangent surface to the same hyperboloid. If we operate a parallel (i.e. maintaining the direction of the shift tangent to the surface of the hyperboloid) transport of this vector along the ϑ, φ = constant surfaces, we shall find that the point O' of maximum approach to the pole O (the origin of the coordinate frame) will have coordinates $(0, a\sin\vartheta\cos\varphi, a\sin\vartheta\sin\varphi)$, hence O' does *not* coincide with O, as it would if the ϑ = constant surfaces were those cones with vertices in O found when the metric is the spherical, and not spheroidal, one. Moreover, we note that the vector $\vec{p}_r$ thus transported along the hyperboloid will in O' be orthogonal to the equatorial coordinate plane. This procedure then shows that there actually *is* an angular momentum with respect to O, whose square modulus will be given by $|p_r|^2a^2\sin^2\vartheta$. This is exactly the term which has come up in (19).

Dealing with hyperbolic orbits, we are then able to physically explain each contribution in equation (15): the value of the separation constant Λ can be set as its value at infinity, i.e.

† Here we should make a brief clarification: at infinity, both $|p_\vartheta|^2 \equiv p_\vartheta p^\vartheta$ and $|p_\varphi|^2 \equiv p_\varphi p^\varphi$, whose physical meaning is that of square of the ϑ and, respectively, azimuthal component of the orbiting particle's *linear* momentum, reduce to zero; what, in contrast, do *not* reduce to zero are p_ϑ^2 and p_φ^2, whose physical meaning is that of square of the ϑ and, respectively, azimuthal component of the particle's *angular* momentum.

the sum of the squared independent contributions to angular momentum, which come not just, as one may obviously argue, from motion in the *angular* variables ϑ and φ, *but also* from the *purely radial* one, which is not so instead. It can also be noted that the radial contribution turns out to be null in three cases only: either with a going to zero (that is, in the spherically symmetric case), or if the particle moves along a parabolic trajectory (zero linear momentum at infinity), and, the third possibility, in the case of stable motion along the symmetry axis.

In summary, we can therefore say that Λ for an unbound orbit is identified as the square modulus of the 'extended' (i.e. also involving pure r-motion contributions) angular momentum, calculated asymptotically.

In the case of bound orbits such asymptotic evaluation cannot of course be made; still, with the reasonable assumption of virialized motion†, we will find the meaning of the 'additional' third term in equation (15) to be physically explainable again. Using the (scalar) virial theorem, this third term becomes

$$2mEa^2 \sin^2 \vartheta = -|p|^2 a^2 \sin^2 \vartheta \tag{20}$$

which is formally analogous to equation (19), *but with a minus sign*. This time, moreover, $|p|^2$ does *not* reduce to the radial part alone, and also the remaining components must be taken into account. As for the radial component, using the same parallel transport as before, we again recognize in its contribution to equation (20) the 'purely radial' squared angular momentum. By the same token, we shall find that the φ-component of equation (20) is the squared angular momentum of the image particle at O' within the $r = 0$ disc, while the ϑ-component gives no contribution. One must, in fact, remember that in this disc, while φ still plays the role of the angular coordinate, ϑ becomes 'the' radial one, hence the motion in ϑ does not engender any angular momentum with respect to O. This effacement of the ϑ-component is, on the other hand, implicit in the structure of the 'additional' term itself; it can, in fact, be thought of as $|\vec{p} \times \vec{a}|^2$, where $\hat{a}$ is the normal versor to the $r = 0$ disc, hence implying the projection of the particle's momentum $\vec{p}$ onto the equatorial coordinate plane. Obviously, this very projection eliminates the ϑ-component of $\vec{p}$, with p_r and p_φ alone surviving to contribute to the 'additional' term we are considering.

We have thus been able to also identify the meaning of the terms defining the third integral of motion Λ for bound virialized orbits. In this case Λ is determined by the sum of the squared 'usual' angular momenta (i.e. those determined by ϑ- and φ-motions) *minus* the squared 'radial' one, *minus* the squared angular momentum of the image particle at O' with respect to O (φ motion contributing alone, within the $r = 0$ disc). It is worth noticing here that Λ can therefore also assume *negative* values, at variance with the unbound orbit case. This also contrasts with what happens in the spherically symmetric case, when the third integral, which is the square modulus of the 'usual' angular momentum, cannot assume negative values‡.

4. Torque in Stäckel potentials

In dealing with axisymmetric potentials many coordinate systems might of course be used; actually, one may profitably employ cylindrical coordinates in modelling lenticular galaxies for instance [1]. Thus proceeding, some interesting features such as regularities in the orbital behaviour may be found, as we have pointed out in the introduction, yet these remain veiled and misty unless a proper choice of coordinates is made. We have, in fact, just seen how the

† This is actually an approximation, since a generic axisymmetric potential, see equation (7), is not a homogeneous function of degree -1; however, its leading term *does* match this requirement.

‡ A naive extension of this property to the axially symmetric situation would therefore have led to a misunderstanding of the properties of Λ: the possibility of negative values would have been lost.

use of an appropriate reference frame is fruitful in providing the requested explanation of the existence and meaning of the third integral in axisymmetric systems. This particular choice of coordinates is, on the other hand, quite natural when dealing with spheroidal gravitational sources ('spheroidal bodies call for spheroidal coordinates', as Binney and Tremaine put it). However, the use of such an advantageous and 'natural' coordinate frame also has some drawbacks, something of the sort of inertial forces which arise in non-inertial frames, although in our case the spurious, i.e. not having a physical origin, effects are not caused by the relative motion of the coordinate frames, but, as we shall now see, by their relative orientation. Since it is not convenient to abandon the otherwise most convenient spheroidal reference frame in order to efface these coordinate effects, we simply have to pay attention to them, so to avoid mistakes in interpreting the results.

Let us therefore consider the Hamilton equations for a particle in a Stäckel potential (12); using spheroidal coordinates (1) they are

$$
\begin{aligned}
&\dot{p}_\varphi = 0 \quad \Longrightarrow \quad p_\varphi = \ell = \text{constant}\\
&\dot{p}_\vartheta = \frac{a^2 \sin\vartheta \cos\vartheta}{\Sigma^2}\left[-(r^2+a^2)p_r^2 - p_\vartheta^2 + \frac{\Sigma^2}{(r^2+a^2)a^2\sin^4\vartheta}p_\varphi^2 - 2m^2 f(r)\right]\\
&\dot{p}_r = \frac{1}{\Sigma^2}\left[ra^2\sin^2\vartheta\, p_r^2 + r p_\vartheta^2 + \frac{r\Sigma^2}{(r^2+a^2)^2\sin^2\vartheta}p_\varphi^2 - m^2\left(\Sigma f'(r) - 2rf(r)\right)\right]
\end{aligned}
\tag{21}
$$

where $f'(r) \equiv \mathrm{d}f(r)/\mathrm{d}r$.

The position (co)vector of the orbiting particle with respect to the origin, which in Cartesian coordinates has components $(r_x, r_y, r_z) = (x, y, z)$, in spheroidal coordinates (1) is given by $(r_r, r_\vartheta, r_\varphi) = (r, a^2 \sin\vartheta \cos\vartheta, 0)$. A torque $\vec{\tau}$ will therefore be measured, acting on the particle's orbit, whose components are given by

$$
\tau^i = \frac{1}{\sqrt{g}}\varepsilon^{ijk} r_j \dot{p}_k \tag{22}
$$

where g is the determinant of the metric (2) and ε^{ijk} is the completely antisymmetric unit tensor; explicitly,

$$
\tau^r = 0 \tag{23a}
$$

$$
\tau^\vartheta = 0 \tag{23b}
$$

$$
\begin{aligned}
\tau^\varphi = {}& \frac{ra^2\cos\vartheta}{\Sigma^3}\left\{-[r^2 + a^2(1+\sin^2\vartheta)]\,p_r^2 - 2p_\vartheta^2 + m^2\left[\Sigma\frac{f'(r)}{r} - 4f(r)\right]\right\}\\
& + \frac{r\cos\vartheta}{(r^2+a^2)^2\sin^4\vartheta}p_\varphi^2.
\end{aligned}
\tag{23c}
$$

Obviously, this torque goes to zero as $r \to \infty$, this being a reflex of the regular behaviour of realistic Stäckel potentials (12) at infinity. Note also the angular dependence, which has interesting implications, as we shall now see.

The existence of the φ-component as the only one different from zero, implies a variation of the Cartesian x- and y-components of the particle's angular momentum; its axial (i.e. the Cartesian z-) component, in contrast, remains constant, in agreement with what is already known, both τ^r and τ^ϑ being null†. This can be shown straightforwardly from the definition

$$J^i = \frac{1}{\sqrt{g}} \varepsilon^{ijk} r_j p_k \tag{24}$$

which leads to

$$\begin{aligned} J^r &= \frac{\ell a^2 \cos\vartheta}{\Sigma} \\ J^\vartheta &= -\frac{\ell r}{\Sigma \sin\vartheta} \\ J^\varphi &= \frac{1}{\Sigma \sin\vartheta}(r p_\vartheta - a^2 p_r \sin\vartheta \cos\vartheta) \end{aligned} \tag{25}$$

or, in Cartesian coordinates

$$J^z = \ell = \text{constant}$$

$$\begin{pmatrix} J^x \\ J^y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} -\dfrac{\ell r \cos\vartheta}{(r^2+a^2)^{1/2}\sin\vartheta} \\ \dfrac{(r^2+a^2)^{1/2}(r p_\vartheta - a^2 p_r \sin\vartheta\cos\vartheta)}{\Sigma} \end{pmatrix}. \tag{26}$$

Equations (26) describe the precession of a vector, whose components vary with r and ϑ, but in such a way as to keep the z-component constant, around the symmetry axis. The existence of this precessional motion of the angular momentum vector is a consequence of the τ^φ term calculated above. The rotation develops in the xy-plane, which is *not* the orbital plane, but the symmetry plane for the coordinates (1) due to the axial symmetry of the metric‡. This implies that not all the calculated torque has a physical origin, since *part* of it is actually induced by our choice of coordinates. This can be better seen if we consider the spherically symmetric ($a \to 0$) limit of equation (23*c*), namely

$$\tau^\varphi_{a=0} = \frac{\ell^2 \cos\vartheta}{r^3 \sin^4\vartheta} \tag{27}$$

correspondingly, we have

$$J^z = \ell = \text{constant}$$

$$\begin{pmatrix} \binom{J^x}{J^y} = \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} -\dfrac{\ell\cos\vartheta}{\sin\vartheta} \\ p_\vartheta \end{pmatrix}. \tag{28}$$

Now, equations (28) seem to set a contradiction with the behaviour of orbits when $a = 0$, i.e. in a spherically symmetric potential: in this case it is well known that they are planar, with a constant angular momentum *vector* (and not just its modulus), while equations (28) indicate a precession of this same vector. Evidently this apparent incongruency is due to the fact that the xy coordinate plane does *not* coincide with the orbital one, hence while the z-component of the particle's angular momentum is still conserved, this does not happen for the rotating

† Attention must be paid in not mistaking τ^φ as a torque giving rise to a rotation around the z-axis: it is a torque *along* the φ-axis, which is everywhere *orthogonal* to the z-axis, hence such a torque engenders *no variation whatever* of the azimuthal angular momentum (which is J^z and not J^φ). In contrast, any non-null τ^r or τ^ϑ would in general have implied such, since both the r and the ϑ axes are not, in general, orthogonal to the z one. Actually, the use of non-Cartesian coordinates might confuse ideas a bit, with respect to the usual way of visualizing things.

‡ Whenever these two planes coincide (a case of stable motion in the equatorial plane) the torque is identically null, as expected.

components in the xy-plane, which are neither null nor even constants, but are functions of the angle ϑ. (However, the same equations (28) imply that the square of the particle's total angular momentum is a constant of motion, which is exactly what one would expect.) When the coordinate equator is made to coincide with the orbital plane, which is the most sensible choice in the spherically symmetric case, then equations (28) show that the precessing components become null (planar motion, at $\vartheta = \pi/2 =$ constant).

It is also worth noticing that equations (28) are also the same in the $a \neq 0$, $r \to \infty$ limit of equations (26); in this case we find a precession effect induced at an infinite distance from the source, whose potential is, on the other hand, null at infinity. Obviously, the further we move from the source, the more its gravitational field resembles that of a monopole, higher multipoles having rapidly decreasing effects, see series (7), and at infinity the coordinate grid coincides with that of the spherically symmetric case. The same comments made just above are therefore still valid: the precession effect vanishes when $\vartheta = \pi/2 =$ constant, for the $a \neq 0$ orbit *at infinity*.

These considerations constitute a couple of enlightening examples of how the results obtained through the use of the advantageous spheroidal coordinates in dealing with spheroidal potentials may be misleading.

When $a \neq 0$ (i.e. when we are interested in studying spheroidal and not spherically symmetric potentials), and at a finite distance from the source, however, the existence of a torque cannot be traced back to coordinate effects only. The study of axisymmetric galactic potentials for instance (see [1]), shows that the orbital plane oscillates (the orbital angular momentum vector precesses around the axis, and its modulus is *not* an integral of motion†) with respect to the (galactic) equatorial plane, due to the non-monopolar structure of the potential, independent of the coordinates used. On the other hand, this is just the kind of orbital behaviour one would expect for a particle moving in a spheroidal-like potential (see [12]). In the $a \neq 0$ case the 'coordinate contribution' to the torque cannot be isolated, in general, since it is mixed up with the 'real' (physical) one in a non-trivial way in equation (23*c*). Yet, we can, under the very plausible condition $a \ll r$, at least isolate what we have above shown to be the pure coordinate-induced torque, through a series development, which leads to

$$\tau^{\varphi} = \tau^{\varphi}_{a=0} + \frac{\cos\vartheta}{a^3}\left\{-2mEa^2\left(\frac{a}{r}\right)^3 + \left[-\frac{2\ell^2\cos^2\vartheta}{\sin^4\vartheta} - \Lambda + 4mEa^2\cos^2\vartheta\right]\left(\frac{a}{r}\right)^5\right\}$$
$$+\frac{m^2\cos\vartheta}{a^3}\left\{af'(r)\left(\frac{a}{r}\right)^4 - 2f(r)\left(\frac{a}{r}\right)^5\right\} + \cdots \qquad (29)$$

where equations (12), (13) and (16) have been used, together with the hypothesis of a regular behaviour of the Stäckel potential (12)‡.

In the $a \neq 0$ terms of equation (29) we are no longer able to separate what is due to the coordinates from what is generated by the actual gravitational field. Still, recalling equation (27), it is interesting to note that the pure coordinate torque $\tau^{\varphi}_{a=0}$ has the same r dependence as the dominating $a \neq 0$ contribution, and that it does indeed constitute a major effect.

† This is obvious, since the modulus of the angular momentum of the orbiting particle is conserved only in central potentials; in a Stäckel potential we have shown that the third integral is *not* the square modulus of the ('usual') angular momentum (except for very special cases, see discussion above, section 3).

‡ This, in particular, means that $f(r)$ must depend on r 'less' than r^2, hence $f'(r)$ 'less' than r. The neglected terms in equation (29) are therefore higher in order than $(a/r)^5$, how much higher depends on the particular form of $f(r)$; the neglected terms not due to $f(r)$ and its derivative are $\mathcal{O}\,(a/r)^7$.

5. Conclusions

In this work our aim has been to analyse some of the consequences that the use of spheroidal coordinates has on the description of axisymmetric gravitational fields. To this end we have focused our attention on the (astrophysically useful) Stäckel potentials, which are those that allow the existence of a third integral of motion (beyond energy and axial angular momentum) for an orbiting particle, through the separation of variables in the Hamilton–Jacobi equation. Some considerations on the structure of the coordinate frame necessary to obtain this separation have led us to understand the physical meaning to be attributed to the third integral; it has also been shown that a distinction between unbound and bound virialized orbits is needed. In the former case, we have seen that the third integral, Λ, is defined by the square modulus of the 'extended' angular momentum of the particle at infinity, this meaning that not only must the 'obvious' ϑ- and φ-contributions be taken into account, but also the *purely radial* one (which is not too obvious, i.e. that a purely radial, ϑ = constant, φ = constant, motion should generate itself an angular momentum). For bound virialized orbits, we have seen that Λ is definable as the *algebraic* sum of the squared contributions to the particle's total angular momentum (which comes not only from the angular motions properly said, but also from the purely radial motion), and of the angular momentum associated with the φ-motion of the particle's image within the $r = 0$ disc. The last two contributions come with the *minus* sign. It follows that *the third integral* Λ *can have negative values too*, in contrast with the corresponding third integral in the spherically symmetric case (to which of course it reduces when $a \to 0$), which instead has properly the meaning of the square modulus (always positive) of the particle's angular momentum; in particular, a negative Λ necessarily implies bound motions.

Together with the analysis of angular momentum topics, we have consequently considered torques in a Stäckel potential. The use of spheroidal coordinates, which are naturally associated with such potentials, determines the existence of a relevant frame-induced contribution to the measured precessional motion of the orbital plane, when this does not coincide with the equator of the reference frame. The obvious way to efface such an effect would of course be that of changing the orientation of the reference frame itself; thus proceeding, however, we would lose the natural correlation of the frame with the characteristics of the source, which is evidently not worthwhile. Then, we just have to remember that a spurious torque is present, which therefore shall not mislead us when analysing the behaviour of the orbit in space.

References

[1] Binney J and Tremaine S 1994 *Galactic Dynamics* (Princeton, NJ: Princeton University Press)
[2] Stäckel P 1895 *C. R. Acad. Sci. Paris* **121** 489
[3] Keres H 1967 *Sov. Phys.–JETP* **25** 504
[4] Israel W 1970 *Phys. Rev.* D **2** 641
[5] Carter B 1966 *Phys. Rev.* **141** 1242
[6] Carter B 1968 *Commun. Math. Phys.* **10** 280
[7] Walker M and Penrose R 1970 *Commun. Math. Phys.* **18** 265
[8] Hughston L P *et al* 1972 *Commun. Math. Phys.* **27** 303
[9] Woodhouse N M J 1975 *Commun. Math. Phys.* **44** 9
[10] Carter B 1977 *Phys. Rev.* D **16** 3395
[11] de Felice F and Preti G 1999 *Class. Quantum Grav.* **16** 2929
[12] Taff L 1985 *Celestial Mechanics* (Chicester: Wiley)